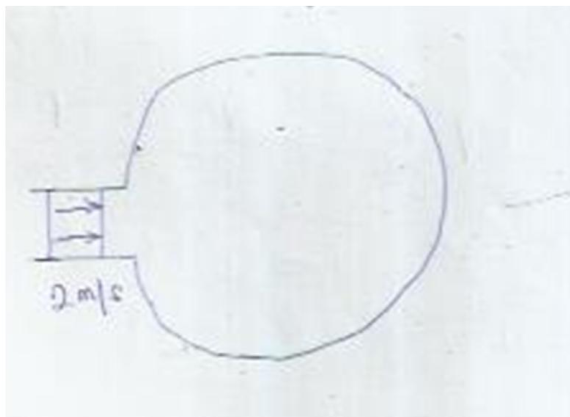
Application of laws governing the motion of bodies to fluid motion

Example 1

A gas flows into a rigid container initially evacuated. Assume that the inflow velocity is uniform at 2mls, as shown below. The tube inlet diameter is 10cm with the volume of the tank equal to 2000 litres. The pressure and temperature in the inlet line are maintained constant at 400kpa and 330K respectively. The gas can be assumed to obey the perfect gas law P = ρRT, with R for the gas equal to 0.30KJ/kg.K. Assume the tank to be non-insulated so that the temperature of the gas in the tank remains constant at a room temperature of 300K. Determine the time required for the pressure in the tank to reach 300kPa.



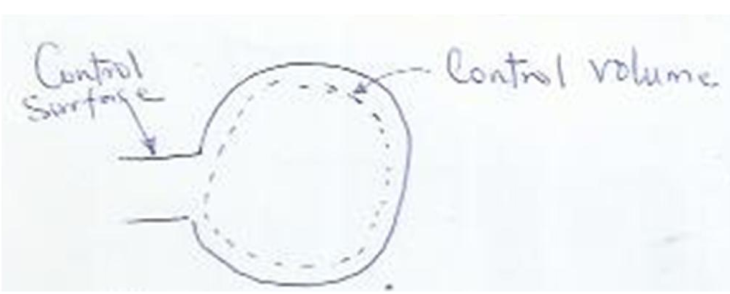
Solution

Flow into the chosen control volume occurs only at the inlet pipe.

From

$$\left.\frac{dM}{dt}\right|_{system} = \left.\frac{\partial x}{\partial t}\right|_{control\,volume} + \int_{cs(inlet\ pipe)} \rho V_n dA = 0$$

$$0 = \left.\frac{\partial x}{\partial t}\right|_{cv} + \int_{(inlet\ pipe)} \rho V_n dA$$



V_n equal 2mls (Efflux is positive) and

$$\rho = \frac{P}{RT} = \frac{400}{0.30\,x\,330} \quad \frac{kN/m^2}{kN.m/kg.K \times K}$$
$$= 4.04kg/m^3$$

Therefore, the second term of the side becomes (4.04) 2dA since density and velocity are constant across the inlet area, they can be taken outside the integral, yielding.

$$(-4.04)(2m/s)x\,\pi/4\,(0.10)^2 = -0.0635kg/s$$
$$therefore,\ \partial m/\partial t = 0.0635kg/s$$
$$\frac{\partial M}{\partial t} = \frac{V\partial\rho}{\partial t} = \frac{V}{Rt}\,x\,\frac{\partial p}{\partial t} \qquad Note: \rho = \frac{p}{RT}$$

Since pressure is only a function of time, we can write the total derivation

$$\frac{V}{RT} \times \frac{dp}{dt} = 0.0635kg/s$$

Integrating, we have

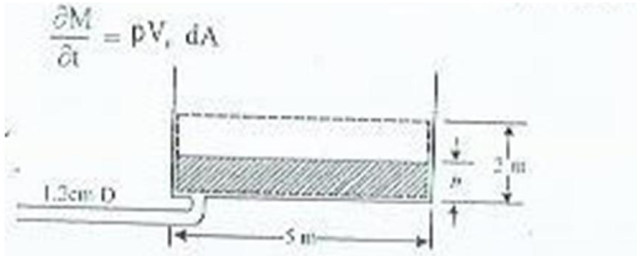
$$\frac{V}{RT}\int_0^{300kpa} dp = 0.0635\int_0^t dt$$

Substituting

$$\frac{2(300)}{0.30\,x\,300}\ \frac{kN/m^2 \times m^3}{kNm/kg.K.K} = 0.0635tKg$$
$$6.7 = 0.0635t\,Kg$$
$$t \approx 105kg$$

Ex 2.

A circular Swimming Pool is 5m in diameter. It is to be filled to a uniform depth of 2m by means of a 1.2cm diameter hose, as shown below. The velocity of the water in the hose is 3mls. Determine the time required to fill the pool in hours.



Solution

Select the control volume to include the entire volume to be filled, as indicate in the diagram

$$0 = \frac{\partial M}{\partial t}\Big|_{cv} + \int_{cv} \ell V_n dA$$

Since the density of water is relatively insensitive to pressure, we have

$$\frac{\partial M}{\partial t} = \ell V_n \, dA$$

The mass of water in the pool at any given time can be expressed as

$$M = \ell \tfrac{\pi}{4} 5^2 h$$

Therefore $\frac{\partial M}{\partial t} = \ell \frac{\pi}{4}\left(5^2\right)\frac{dh}{dt} = \ell V_n A$

Or

$$\frac{dh}{dt} = \frac{3 \; x \; \pi/4 \; x \; 0.01^2 \; x \; 2}{\pi/4 \; x \; 5^2} = 17.28 \, x \, 10^{-6} \, m/s$$

Integrating, we have

$$\int_0^{2m} dh = 0.00001728 \int_0^t dt$$

$$t = \frac{2}{2 \; x \; 10^{-5}}$$

$$= 1157415$$
$$= 32.15 hrs.$$

VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

Velocity Potential function: It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by $\phi(phi)$. Mathematically, the velocity potential is defined as $\phi = f(x, y, z)$ for steady flow such that:

$$\mu = \frac{-\partial\phi}{\partial x}, v = \frac{-\partial\phi}{\partial y}, w = \frac{-\partial\phi}{\partial z}$$

u, v, w are the components of velocity in x, y, z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms fo velocity potential function are given by:

$$u_r = \frac{\partial\phi}{\partial r}, u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

U_r = Velocity component in radial (r) direction

U_θ= Velocity component in tangential direction (θ direction)

The continuity equation for an incompressible steady flow is

$$\frac{-\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substituting the values of u, v and w from equation above , we get

$$\frac{\partial}{\partial x}\left(\frac{-\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{-\partial\phi}{\partial z}\right) = 0$$

Or $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \{Laplace equation\}$

For two-dimension case the equation reduce to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Properties of the potential function

The rotational components* are given by

$$w_z \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$w_y =\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$w_x =\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from

$$u=\frac{-\partial v}{\partial x}, v=\frac{-\partial \phi}{\partial y}, w=\frac{-\partial \phi}{\partial z}$$

$$w_z =\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{-\partial \phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]=\frac{1}{2}\left[\frac{\partial_2 \phi}{\partial x \partial y}+\frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$w_y =\frac{1}{2}\left[\frac{\partial}{\partial z}\left(\frac{-\partial \phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]=\frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x}+\frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$w_x =\frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{-\partial \phi}{\partial z}\right)\right]-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)=\frac{1}{2}\left[\frac{-\partial^2 \phi}{\partial y \partial z}+\frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\dfrac{\partial^2 \phi}{\partial x \partial y}=\dfrac{\partial^2 \phi}{\partial y \partial x}$

then $=\dfrac{\partial^2 \phi}{\partial z \partial x}=\dfrac{\partial^2 \phi}{\partial x \partial z}; etc$

$$\therefore w_x = w_y = w_x = 0$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are:

1. If velocity potential ϕ exists, the flow should be irrotational
2. If velocity potential ϕ (satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function

It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (Psi) and defined only for two dimensional flow.

Mathematically, for steady flow it is defined as ψ =f(x,y) such that

$$\frac{\partial\psi}{\partial x} = v \, and \, \frac{\partial\psi}{\partial y =} - u$$

The velocity component in cylindrical polar co-ordinates in terms of stream function are given as

$$U_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi} and \, U_\theta = -\frac{\partial\psi}{\partial r}$$

U_r=radial velocity and U_θ= tangential velocity

The continuity equation for two-dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the value of u and v in

$$\frac{\partial\psi}{\partial x} = v, \frac{\partial\psi}{\partial y} = -u \, we \, have$$

$$\frac{\partial}{\partial x}\left(\frac{-\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial\psi}{\partial x}\right) = 0 \, or \, \frac{-\partial^2\psi}{\partial x\partial y} + \frac{-\partial^2\psi}{\partial x\partial y} = 0$$

Hence existence 0 φmeans a possible case of fluid flow. The fluid may be rotational or irrotational

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{-\partial z}{\partial x} - \frac{-\partial u}{\partial y}\right)$ Substituting the values of u and v above equation, we have

$$\omega_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \left(\frac{-\partial\psi}{\partial x}\right) - \frac{-\partial}{\partial y}\left(\frac{-\partial\psi}{\partial y}\right)\right)$$

$$= \frac{1}{2}\left[\frac{-\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]$$

For irrotational flow, ω_z=0. Hence above equation becomes $\frac{-\partial^2\psi}{\partial x^2}+\frac{-\partial^2\psi}{\partial y^2}=0$

Which is Laplace equation for ψ

The properties of stream function ψ are:

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational
2. If stream function (ψ) satisfies Laplace equation, it is a possible case of an irrotational flow

Equipotential line: A line along which the velocity potential ϕ is constant is called equipotential line.

For equipotential line ϕ = Constant

$$\partial\psi = 0$$

$$\phi = f(x, y)\, for\, steadly\, flow$$

But
$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$
$$= -udx - vdy$$
$$= -(udx + vdy)$$

For equipotential line, $d\phi = 0$

$$-(udx + vdy) = 0\, or\, udx + vdy = 0$$
$$\therefore \frac{\partial y}{\partial x} = \frac{-u}{v}\, but\, \frac{\partial y}{\partial x} = slope\, of\, equipotential\, line$$

Line of constant stream function

ψ =constant

$d\psi$=0

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = +vdx - udy`$$

For a line of constant stream function

$$= d\psi = 0 \, or \, vdx - udy = 0$$

$$\frac{\partial y}{\partial x} = \frac{v}{u} but \frac{\partial y}{\partial x} is\, slope\, of\, stream\, line$$

From equation $\frac{\partial y}{\partial x} = \frac{-u}{v}$ (slope of equipotential line) and $\frac{\partial y}{\partial x} = \frac{u}{v}$ (slope of stream line). It is clear that the product of slope of equipotential line and the slope of stream line at the point of intersection is equal to-1. Thus the equipotential lines are orthogonal to the stream lines at all points of intersection.

Relationship between stream Function and velocity potential function.

From u= - $\frac{\partial \phi}{\partial x} = and\ v = \frac{\partial \phi}{\partial y} and$

$$u = -\frac{\partial \psi}{\partial y} and\ v = \frac{\partial \psi}{\partial x}$$

Thus we have $u = \frac{-\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial y} and$

$$v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} and \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}$

Example 1

The velocity potential function is given by an expression $\phi = \frac{-xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$

(i) Find the velocity components in x and y directions

(ii) Show that ϕ represents a possible case of flow

Given $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$

The partial derivations of ϕ w.r.t to x and y are:

$$\frac{\partial \phi}{\partial x} = \frac{-y^3}{3} - 2x + \frac{3x^2 y}{3}$$

$$and\ \frac{\partial \phi}{\partial y} = \frac{-3xy^2}{3} + \frac{x^3}{3} + 2y$$

(1) The velocity components u and v are given

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}, w = \frac{-\partial \phi}{\partial z}$$

$$\Rightarrow u = \frac{-\partial \phi}{\partial x} = -\left[\frac{y^3}{3} - 2x + \frac{3x^2 y}{3}\right]$$

$$= \frac{y^3}{3} + 2x \frac{-3x^2 y}{3}$$

$$= \frac{y^3}{3} + 2x - x^2 y$$

$$v = \frac{\partial \phi}{\partial y} = -\left\{\frac{-3xy^2}{3} + \frac{x^3}{3} + 2y\right\}$$

$$= xy^2 - x^3/3 - 2y$$

(iii) The given value of ϕ , will represent a possible case of flow if it satisfies the Laplace equation i.e. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

But $\frac{\partial \phi}{\partial x}$ is solved to be $\frac{-y^3}{3} - 2x + x^2 y$

And $\frac{\partial \phi}{\partial y}$ is solved to be $-xy^2 + \frac{x^3}{3} + 2y$

$$\therefore if\ \frac{\partial \phi}{\partial x} = \frac{-y^3}{3} - 2x + x^2 y$$

$$\frac{\partial^2 \phi}{\partial x^2} = -2 + 2xy$$

$$and\ if\ \frac{\partial \phi}{\partial y} = -xy^2 + \frac{x^3}{3} + 2y$$

$$\frac{\partial^2 \phi}{\partial y^2} = -2xy + 2$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (-2 + 2xy) + (-2xy + 2)$$

$$= -2 + 2xy - 2xy + 2$$

$$= 0$$

:.Laplace equation is satisfied and hence represent a possible case of flow.

Example 2:

The velocity potential function is given by $\phi = 5(x^2 - y^2)$ Calculate the velocity components at the point (4,5)

Solution

$$\phi = 5(x^2 - y^2)$$

$$\frac{\partial \phi}{\partial x} = 10x\ and\ \frac{\partial \phi}{\partial y} = -10y$$

By velocity components u and V are given as: $u = -\frac{\partial \phi}{\partial x} = -10x\ and\ v = -\frac{\partial \phi}{\partial y} = 10y$

The velocity components at the point (4,5) i.e., at x=4, y=5

U = -10 x 4 = -40 unit

V = 10 x 5 = 40 unit

Problem 3

A stream function is given by $\psi = 5x - 6y$ calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

$\psi = 5x - 6y$

$\frac{\partial \psi}{\partial x} = 5 \, and \, \frac{\partial \psi}{\partial y} = -6$

But the velocity components u and v in terms of stream function are given as

$$u = -\frac{\partial \psi}{\partial y} andv = +\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial \psi}{\partial x} = 5units/\sec andu = -\frac{\partial \psi}{\partial y} = 6uits/\sec$$

Resultant Velocity = $= \sqrt{u^2 + u^2}$

$$= \sqrt{6^2 + 5^2}$$

$$= \sqrt{61}$$
$$= 7.91unit/\sec$$

Direction is given by, $\tan = \theta = \frac{v}{u}$

$$\tan\theta = 5/6 = 0.8333$$
$$\theta = \tan^{-1}(0.8333) = 39^0 \, 48^1$$